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The minimax method is applied to boundary value problems which arise in structural mechanics. This is a weighted residual method in which the maximum absolute value of a residual is minimized. Like other weighted residual methods, a trial function is employed which consists of undetermined parameters and basic functions. This trial function is introduced into governing differential equations, and the maximum absolete residual among several residuals at discrete points in the domain is minimized. This residual minimization criterion is applied to a finite element formulation by using piecewise trial functions defined on each element.

THE MINIMAX FINITE ELEMENT METHOD

by

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INTRODUCTION

In this paper, the minimax method is applied to boundary value problems which arise in structural mechanics. This is a weighted residual method in which the maximum absolute value of a residual is minimized. Like other weighted residual methods (7, 13), a trial function is employed which consists of undetermined parameters and basis functions. This trial function is introduced into governing differential equations, and the maximum absolute residual among several residuals at discrete points in the domain is minimized. This residual minimization criterion is applied to a finite element formulation by using piecewise trial functions defined on each element.

Computational implementation is achieved using linear programming. This approach has been used in the global sense to obtain solutions for differential equations (3, 18, 21, 28).

Since the linear programming technique can be applied to solve overdetermined systems of equations (24), more mesh points than the number of unknown parameters can be used to improve the solution. This is in contrast to the collocation method. Also, equality constraints representing the boundary conditions and inter-element continuity conditions can be included in the formulation. This feature provides more freedom in choosing a trial function, which is often the most important step in the weighted residual method.

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In the collocation method, the roots of the orthogonal polynomial are frequently used as collocation points with considerable success (16, 23,26,27). This is referred to as the orthogonal collocation method. Convergence studies (8,10,12,20) of the orthogonal collocation method show that in most cases a convergence rate similar to that for the Galerkin and Ritz methods can be obtained. This procedure will be utilized here in choosing optimal locations for mesh points.

Since inequality constraints are acceptable in a linear programming problem, the minimax method can be used for limit analyses and can analyse structures with off-set supports, which pose a contact problem. When multiple optimal feasible solutions arise for these types of problems, a parametric programming method described in Appendix I can be used to choose a specific optimal solution.

Soon after the upper bound theorem and the lower bound theorem in plasticity became available (19) and the linear programming technique was developed by Dantzig, limit analysis was identified as a linear programming problem, and linear programming algorithms have been used for limit analysis solutions since then (1,4,5,10,15,17,29,30).

This method appears to be easy to formulate and simple to use. Like the collocation method, the proposed method does not involve the integration that is necessary with the other weighted residual methods.

FORMULATION

Elasto-Static Analyses.- For a given governing differential operator equation of motion

$$Au = f \tag{1}$$

we seek an approximate solution $\tilde{u}$ whose image under differential operator A approximates the given function f. Let

$$\tilde{u}(\underline{x}) = \sum_{i=1}^{I} a_i \phi_i (\underline{x}) \tag{2}$$

in which $\underline{x}$ is a vector of space variables, $\tilde{u}(\underline{x})$ is a trial function, a_i are unknown parameters, and $\phi_i(\underline{x})$ are bases of the trial function. Introduce Eq. (2) into Eq. (1). The residual $R(\underline{x})$ becomes

$$R(\underline{x}) = A\tilde{u}(\underline{x}) - f(\underline{x}) = \sum_{i=1}^{I} a_i \, [A\phi_i(\underline{x})] - f(\underline{x}) \tag{3}$$

In the proposed method, the criterion to determine the unknown parameters a_i is to minimize the maximum absolute residual among several residuals at discrete points. Let

$$r = \max|R(\underline{x}_j)| \qquad j = 1,2,\ldots J \tag{4}$$

in which $\underline{x}_j$ are the locations of mesh points. Then, a linear programming problem can be established (24). Find the parameters a_i such that $Z = r$ is minimized and the constraints

$$\left.\begin{array}{l} R(\underline{x}_j) - r \leq 0 \\ -R(\underline{x}_j) - r \leq 0 \end{array}\right. \qquad j = 1,2,\ldots J \tag{5}$$

are satisfied.

If the trial function does not satisfy some of the boundary conditions, the following equality constraints can be included in the above formulation (Eq. 5).

$$Bu(\underline{x}_p) = b(\underline{x}_p) \qquad p = 1,2,\ldots P \tag{6}$$

in which B is a boundary differential expression, $b(\underline{x})$ is a given function defined on the boundary, and $\underline{x}_p$ are the locations of boundary mesh points.

When the finite element method is used, interelement continuity conditions should be satisfied. That is, if p is the number of the highest order appearing in the governing differential equation, the approximate function and up to (p-1) derivatives must be continuous across interelement boundaries (31). In the minimax method, these conditions can be placed as constraints in the linear programming formulation. However, the high order inter-element continuity requirement tends to increase the problem size by increasing the number of nodal variables or by using additional constraint equations. This constitutes a drawback of the proposed minimax method compared to conventional finite element methods.

If the number of constraints is greater than or equal to the number of unknown parameters, the existence and uniqueness of a polynomial of best approximation is guaranteed (9, 22).

Limit Analysis.- According to the lower bound theorem in plasticity, if an equilibrium distribution of stress can be found which balances the applied load and is everywhere below yield or at yield, the structure will not collapse or will be just at the point of collapse (19).

To apply the previously formulated minimax method to limit analysis by using the lower bound theorem, the trial function should satisfy the equilibrium equations in the domain and boundary conditions on the boundary. The pattern of load distribution is known. However, the magnitude of the load factor λ is unknown and is treated as an unknown variable together with unknown coefficients of a trial function. The yield condition is introduced

at some check points as inequality constraints. The minimization of the maximum residual produces only an equilibrium state for any value of λ. In other words, for any λ which does not violate the yield condition, the values of unknown coefficients can be determined for the resulting stress distribution to balance the applied load, λ, accordingly. This means that the linear programming problem has multiple solutions unless there is another constraint that can specify λ. Our purpose is to obtain a solution that has the maximum value for λ among these multiple solutions. Therefore, we have to minimize the maximum residual and, at the same time, maximize λ. Then, the linear programming becomes: find the a_i such that $Z = r$ is minimized, λ is maximized, and the constraints

$$\begin{array}{ll} R(\underline{x}_j) - r \le 0 & \\ & j = 1,2,\ldots J \\ -R(\underline{x}_j) - r \le 0 & \\ Bu(\underline{x}_p) = b(\underline{x}_p) & p = 1,2,\ldots P \\ Y(\underline{x}_n) \le k & n = 1,2,\ldots N \end{array} \tag{7}$$

are satisfied. Here $Y(\underline{x}_n)$ is the value of the yield function evaluated at $\underline{x} = \underline{x}_n$.

This problem of multiple objective functions can be handled as a linear programming problem using the parametric programming method described in Appendix I. The technique in Appendix I of obtaining a solution with a maximum λ is to use a new objective function $Z = r - \varepsilon\lambda$, where ε is a small positive number. The solution gives a lower bound to the limit load.

NUMERICAL EXAMPLES AND RESULTS

Beam on Elastic Foundation.- Consider a beam on an elastic foundation subject to a uniform load as shown in Fig. 1.

The governing differential equation for engineering beam theory is

$$EIy^{iv} + k_s y = q \tag{8}$$

in which E is Young's modulus of elasticity, I is the moment of inertia of the beam cross section, k_s is the stiffness of the elastic foundation, and q is the uniform load intensity.

First, an ordinary polynomial is used as a trial function.

$$\tilde{y}(x) = \sum_{i=1}^{I} a_i \, x^{i-1} \tag{9}$$

Then,

$$R(x) = EI\tilde{y}^{iv}(x) + k_s \tilde{y}(x) - q \tag{10}$$

Let one element apply for the entire beam. Use an equidistant spacing mesh for the numerical calculations. The results shown in Table 1 are quite accurate.

An exception occurs for the beam of length L = 2 with a 10th order polynomial. In the use of a very high order polynomial, the differences between magnitudes of the coefficients in the constraint equations become very big and significant numerical error results. This is a drawback in using an ordinary polynomial. However, this difficulty can be overcome by refining elements. An advantage of using an ordinary polynomial is that there is no need for matrix inversion to transform the coefficients of the polynomial to nodal variables as is encountered in the usual finite element formulations (31).

Next use Hermitian shape functions as bases of the trial function. Let $H^n_{ij}(\xi)$ denote the nth order Hermitian shape function which is the (2n+1)th order polynomial (31). In this polynomial, n is the number of derivatives that the set can interpolate, i is the order of derivatives of $H^n_{ij}(\xi)$ with respect to ξ, and j = 1 or 2 are the element node numbers. In the numerical example, $H^2_{ij}(\xi)$ and $H^3_{ij}(\xi)$ are used. Then, the trial function for each

element becomes

$$\tilde{y}_e = \sum_{j=1}^{2} (H^2_{0j}(\xi) y_j + H^2_{1j}(\xi)\ y_{,\xi j} + H^2_{2j}(\xi)\ y_{,\xi\xi j}) \qquad (11)$$

or

$$\tilde{y}_e = \sum_{j=1}^{2} (H^3_{0j}(\xi)\ y_j + H^3_{1j}(\xi)\ y_{,\xi j} + H^3_{2j}(\xi)\ y_{,\xi\xi j} + H^3_{3j}(\xi)\ y_{,\xi\xi\xi j}) \qquad (12)$$

respectively.

As shown in Table 2, the numerical results are quite accurate and, for the same number of unknowns, increasing the number of mesh points improves the solution. It is also noted that improvement can be achieved by refining elements.

When there is a concentrated load, the structure can be divided into finite elements such that the location of the concentrated load becomes a node point. The discontinuity of a variable such as y'" should be taken into account in the formulation. This technique is also used in the analysis of beams with off-set supports.

Torsion of Prismatic Bar. - As an application of the minimax method to partial differential equations consider the elastic torsion of a prismatic bar. The governing differential equation (25) is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta \quad \text{in the domain} \qquad (13)$$

$$\phi = 0 \quad \text{on the boundary} \qquad (14)$$

in which ϕ is Prandtl's stress function, G is the shear modulus, and θ is the angle of twist per unit length of the bar.

Since the highest order derivative appearing in the governing differential equation is two, the requirement of interelement continuity is the continuity of ϕ, $\phi_{,x}$, $\phi_{,y}$, and $\phi_{,xy}$. Therefore, the following trial function can be used for the rectangular element shown in Fig. 2.

$$\tilde{\phi}_e(\xi,\eta) = \sum_{i=1}^{2}\sum_{j=1}^{2} [H^1_{0i}(\xi)\ H^1_{0j}(\eta)\ \phi_{ij} + H^1_{1i}(\xi)\ H^1_{0j}(\eta)\ \phi_{,\xi ij}$$

$$+ H^1_{0j}(\xi)\ H^1_{1j}(\eta)\ \phi_{,\eta ij} + H^1_{1i}(\xi)\ H^1_{1j}(\eta)\ \phi_{,\xi\eta ij}]$$

The residual in each element is

$$R_e(\xi,\eta) = \frac{1}{a^2}\frac{\partial^2\tilde{\phi}_e}{\partial\xi^2} + \frac{1}{b^2}\frac{\partial^2\tilde{\phi}_e}{\partial\eta^2} + 2G\theta \qquad (15)$$

As a numerical example, a prismatic bar of a square cross section will be treated by the minimax method. Let H be the height and W the width of the cross section. The analytic solution for the torsion of a rectangular bar as taken from Ref. 25 is

$$\tau_{max} = KG\theta W$$

$$M_t = K_1 G\theta W^3 H$$

in which τ is shear stress, M_t is the torque, and K and K_1 are numerical factors depending on the ratio H/W.

Due to the symmetry, only one quadrant of the cross section needs to be considered. Results are shown in Table 3. For the same number of mesh points, the selection of mesh points at the Gaussian points gives better results than the use of mesh points at a/4 and 3a/4. It is also noted that the solution for K with 3x3 Gaussian points is better than with 2x2 Gaussian points. However, 2x2 Gaussian points plus a center mesh point gives poorer results than simple 2x2 Gaussian mesh points. This indicates that convergence with respect to increasing mesh points is not monotonic. The results also indicate that increasing the number of elements improves the results even with a modicore choice of mesh points.

Plane Stress Analysis.- For a plane (x,y) elasticity problem the equilibrium equation is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad (16)$$

if body forces and thermal effects are neglected. In terms of displacement components u and v , this becomes

$$\frac{1}{1-\nu} u,_{xx} + \frac{1+\nu}{2(1-\nu)} v,_{xy} + \frac{1}{2} u,_{yy} = 0$$

$$\frac{1}{1-\nu} v,_{yy} + \frac{1+\nu}{2(1-\nu)} u,_{xy} + \frac{1}{2} v,_{xx} = 0 \qquad (17)$$

The solution to Eqs (17) subject to appropriate boundary conditions constitutes the solution of the problem of elasticity. Since the highest order of derivatives appearing in the governing differential equations is two in Eq. (18), u, $u,_x$, $u,_y$, and $u,_{xy}$ should be continuous between elements. The

same applies to the displacement component v. Thus, a conforming element using the Hermitian shape functions H^1_{ij} as basis functions is used.

$$\tilde{u}_e = \sum_{i=1}^{2} \sum_{j=1}^{2} [H^1_{0i}(\xi)\, H^1_{0j}(\eta)\, u_{ij} + H^1_{1i}(\xi)\, H^1_{0j}\, u_{,\xi ij}$$

$$+ H^1_{0j}(\xi)\, H^1_{1j}(\eta)\, u_{,\eta ij} + H^1_{1i}(\xi)\, H^1_{1j}(\eta)\, u_{,\xi\eta ij}]$$

$$\tag{18}$$

$$\tilde{v}_e = \sum_{i=1}^{2} \sum_{j=1}^{2} [H^1_{0i}(\xi)\, H^1_{0j}(\eta)\, v_{ij} + H^1_{1i}(\xi)\, H^1_{0j}(\eta)\, v_{,\xi ij}$$

$$+ H^1_{0i}(\xi)\, H^1_{1j}(\eta)\, v_{,\eta ij} + H^1_{1i}(\xi)\, H^1_{1j}(\eta)\, v_{,\xi\eta ij}]$$

in which $\tilde{u}_e$, $\tilde{v}_e$ are trial functions for u and v, respectively, in an element.

As an example of a plane stress problem, consider a plate subject to simple in-plane forces such as pure tension, pure bending, and pure shear. For the pure tension problem shown in Fig. 3, the approximate boundary conditions are

$$u_1 = u_2 = u_3 = u_{,y1} = u_{,y2} = u_{,y3} = u_{,y4} = u_{,y7}$$

$$= v_1 = v_4 = v_7 = v_{,x1} = v_{,x2} = v_{,x3} = v_{,x4} = v_{,x7} = 0$$

$$\sigma_{xx7} = \sigma_{xx8} = \sigma_{xx9} = 10$$

$$\sigma_{yy3} = \sigma_{yy6} = \sigma_{xy6} = \sigma_{xy8} = \sigma_{xy9} = 0$$

in which u_1 is the value of u at node 1, $u_{,y1}$ is the value of $\frac{\partial u}{\partial y}$ at node 1, σ_{xx7} is the value of σ_{xx} at node 7, etc.

The results obtained with the minimax method for the simple in-plane forces are exact and this formulation passes the so-called patch test. According to this test, in order for a solution to converge to the correct one by refining elements, a patch of elements subjected to a specific nodal displacement corresponding to a state of constant strain should produce the constant strain state throughout the elements (6).

Limit Analysis of a Fixed-fixed Beam.- Suppose the limit load is sought for a prismatic fixed-fixed beam subject to a uniform load as shown in Fig. 4. Due to the symmetry, only half of the beam needs to be considered.

Let

$$\tilde{y}(\xi) = \sum_{i=1}^{2} [H^3_{0i}(\xi)\, y_i + H^3_{1i}(\xi)\, y_{,\xi i} + H^3_{2i}(\xi)\, y_{,\xi\xi i} + H^3_{3i}(\xi)\, y_{,\xi\xi\xi i}] \quad (19)$$

For a beam problem, the yield condition is expressed as

$$|M_n| \leq M_p \quad (20)$$

in which M_n is the moment at $x = x_n$ and M_p is the plastic yield moment of the beam which is obtained when the whole section of the beam becomes plastic (2), as shown in case C in Fig. 5. If the yield condition is checked at $\xi = 0$, $\xi = 1$

$$M_n = -\frac{EI}{\ell^2} y_{,\xi\xi n} \qquad n = 1, 2 \quad (21)$$

Choose the uniformly distributed load $q = 1$. Then, the residual is

$$R(\xi) = \frac{d^4\tilde{y}}{d\xi^4} - \frac{\lambda \ell^4}{EI} \quad (22)$$

The linear programming problem becomes: Find y_i, $y_{,\xi i}$, $y_{,\xi\xi i}$, $y_{,\xi\xi\xi i}$ such that $Z = r$ is minimized, λ_1 is maxmimized, and the constraints

$$\begin{aligned} R(\xi_j) - r &\leq 0 \\ -R(\xi_j) - r &\leq 0 \\ \left|-\frac{EI}{\ell^2} y_{,\xi\xi 1}\right| &\leq M_p \\ \left|-\frac{EI}{\ell^2} y_{,\xi\xi 2}\right| &\leq M_p \end{aligned} \quad (23)$$

are satisfied. Using ξ_j = 0.2, 0.4, 0.6, 0.8, and 1, the following results were obtained

M_1 = -12 lb-in (1356.36 N-mm), V_1 = 7.2 lb (32.04 N), y_2 = 0.004 in (0,1016 mm)

M_2 = 6 lb-in (678.18 N-mm), λ_1 = 1.44 lb (6.408 N)

Here, M_i and V_i are the moments and shear force at node i. λ_1 = 1.44 is the

lower bound on the limit load factor and causes the plastic moment at $\xi = 0$.

If the material is elastic perfectly plastic, M_1 remains the same as M_p. Upon a further increase in load, the elastic part of the beam will support the increase in load. Therefore, by solving the following linear programming problem, the next largest load which causes further yielding can be determined: Find y_i, $y,_{\xi i}$, $y,_{\xi\xi i}$, and $y,_{\xi\xi\xi i}$ such that $Z = r$ is minimized, λ_2 is maximized, and the constraints

$$
\begin{aligned}
&R(\xi_j) - r \leq 0 \\
&-R(\xi_j) - r \leq 0 \\
&\left| -\frac{EI}{\ell^2} y,_{\xi\xi 2} \right| < M_p \\
&-\frac{EI}{\ell^2} y,_{\xi\xi 1} = M_p
\end{aligned}
\qquad (24)
$$

are satisfied. The numerical results for this problem were found to be

$M_1 = -12$ lb-in (-1356.36 N-mm) $V_1 = 9.6$ lb (42.72 N), $y_2 = 0.11$ in. (2.794 mm)

$M_2 = 12$ lb-in (1356.36 N-mm) $\lambda_2 = 1.92$ lb (8.544 N)

As indicated in Fig. 4(d), the collapse mechanism has been formed and $\lambda_2 = 1.92$ is the exact load factor for the collapse load.

CONCLUSION

It is shown that the minimax weighted residual method can be used for obtaining finite element solutions. The method appears to be relatively easy to set up and gives satisfactory results for the example problems. This method is very attractive, particularly for ordinary differential equations and low order partial differential equations. It can be used to solve problems with inequality constraints.

Unlike the collocation method, the solution can be improved by using more mesh points than the number of unknown coefficients in the given trial function. Also, the solution can be improved by refining elements. The Gaussian points are optimal mesh points for the proposed minimax method.

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APPENDIX I - PARAMETRIC PROGRAMMING METHOD TO CHOOSE A PARTICULAR SOLUTION FROM MULTIPLE SOLUTIONS

Consider the problem, find $\{x\}$ such that $Z = \{c\}^T\{x\}$ are minimized and the constraints

$$[A]\{x\} = \{b\}$$

$$\{x\} \geq 0$$

are satisfied. Let some perturbation be given to c

$$\{c^+\}^T = \{c\}^T - \varepsilon\{f\}^T$$

in which ε is small positive number. Now, assume that $\{x_B^q\}^T$ is one of the multiple solutions $\{x_B\}$. Then, Z^+ becomes (14)

$$Z^+ = (\{c_B^q\}^T - \varepsilon\{f_B^q\}^T)\ \{x_B^q\} = \{c_B^q\}^T\ \{x_B^q\} - \varepsilon\{f_B^q\}^T\ \{x_B^q\}$$

If there are ℓ multiple solutions

$$\{c_B^1\}^T\ \{x_B^1\} = \{c_B^2\}^T\ \{x_B^2\} = \ldots\ldots = \{c_B^q\}^T\ \{x_B^q\} = \ldots\ldots = \{c_B^\ell\}^T\ \{x_B^\ell\}$$

If we can have one of x_B remain optimal, minimizing Z^+ means also maximizing $\varepsilon\{f_B\}^T\ \{x_B\}$ and this is the same as

$$\text{Maximize } Z^* = \varepsilon\{f\}^T\ \{x\}$$

$$\text{such that } [A]\{x\} = \{b\}$$

$$\{x\} \in \{x_B\}$$

Therefore, by minimizing $Z^+ = \{c^+\}^T\{x\}$ such that $[A]\{x\} = \{b\}$, $\{x\} \geq 0$ and by properly choosing ε, we can obtain a particular solution from the multiple solutions which satisfy min $Z = \{c\}^T\{x\}$ and max $Z^* = \varepsilon\{f\}^T\{x\}$ such that $[A]\{x\} = \{b\}$, $\{x\}\in\{x_B\}$, ε can be chosen as follows: We wish to maintain one of the $\{x_B\}$ as the optimal basic solution for the new problem

$$\text{minimize } Z^{+} = \{c^{+}\}^{T}\{x\}$$

$$\text{such that } [A]\{x\} = \{b\}$$

$$\{x\} \geq 0$$

Denote by $z_j^+ - c_j^+$ the value at $z_j - c_j$ when $\{c\}^T$ is replaced by $\{c^+\}^T$, then the critical value of ε is such that any increase in ε would make one or more $z_j^+ - c_j^+$ positive

$$z_j^+ - c_j^+ = (\{c_B\}^T - \varepsilon\{f_B\}^T)\{y_j\} - c_j - \varepsilon f_j = z_j - c_j - \varepsilon(\{f_B\}^T\{y_j\} - f_j)$$

in which f_B is the row vector that contains the components of f corresponding to the components of c in c_B. If $\{f_B\}^T\{y_j\} - f_j$ are nonnegative, then we can make ε arbitrarily large without destroying optimality. However, if one or more $\{f_B\}^T\{y_j\} - f_j$ are negative and ε is large enough, the corresponding $z_j^+ - c_j^+$ will become positive. Thus, the critical value of $\varepsilon_c \geq 0$ is given by

$$\varepsilon_c = \min \frac{z_j - c_j}{\{f_B\}^T\{y_j\} - f_j}, \text{ if } \{f_B\}^T\{y_j\} < 0 \text{ for one or more } j$$

or

$$\varepsilon_c = \infty, \text{ if } \{f_B\}^T\{y_j\} - f_j \geq 0 \text{ , for all } j$$

Therefore, by choosing $\varepsilon \leq \varepsilon_c$, minimizing $Z^+ = \{c^+\}^T\{x\}$ also minimizes $Z = \{c\}^T\{x\}$ and, at the same time, maximizes $z^* = \varepsilon\{f\}^T\{x\}$.

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APPENDIX III. - NOTATION

The following symbols are used in this paper:

A = differential operator;

$[A]$ = constraint matrix;

a_i = unknown coefficients;

$\{a_j\}$ = columns of $[A]$ matrix;

B = boundary differential expression;

$[B]$ = basis matrix;

$\{c\}^T$ = row price vector;

$\{c_B\}^T$ = row vector of the prices of basis variables;

E = Young's modulus of elasticity;

f = prescribed function;

$\{f\}^T$ = some specific, but arbitrary row vector;

G = shear modulus;

H_{ij}^n = nth order Hermitian shape function;

I = moment of inertia of cross section;

k_s = stiffness of elastic foundation;

L = length of a beam;

ℓ = length of a beam element;

M_i = moment at node i;

M_p = plastic moment of a beam;

M_t = torque;

q = uniform load intensity;

R = residual;

r = maximum of the absolute value of residual ;

u = exact solution;

$\tilde{u}$ = approximate solution (trial function);

V_i = shear at node i ;

$\underline{x}$ = vector of space variables;

Y = yield function;

y_j = value of y at node j;

$\{y_j\}$ = $[B^{-1}]\{a_j\}$;

$y_{,\xi j}$ = value of $\frac{\partial y}{\partial \xi}$ at node j;

Z = objective function;

Z_1 = secondary objective function;

z_j = $\{c_B\}^T \{y_j\}$;

$\phi_i(x)$ = basis of a trial function;

ϕ = Prandtl's stress function;

ϕ_{ij} = value of ϕ at node (i,j) (Refer to Fig. 3);

$\phi_{,\xi ij}$ = value of $\frac{\partial \phi}{\partial \xi}$ at node (i,j);

λ = load factor;

ν = Poisson's ratio; and

θ = angle of twist per unit length.

TABLE 1 Results for a Beam on Elastic Foundation Subject to Uniform Load by using an Ordinary Polynomial

Beam length (ft) (1)	Degree of poly-nomial (2)	No. of unknowns (3)	No. of B.C. (4)	No. of mesh points (5)	No. of con-straints (6)	Error (%) at locations		
						x = L/20 (7)	x = L/2 (8)	x = 19 L/20 (8)
2	5	6	4	3	7	0.03	0.00	0.04
2	10	11	4	7	11	-0.04	-0.48	6749.24
1	5	6	4	3	7	0.00	0.00	0.00
1	10	11	4	7	11	0.01	0.00	0.01

Note: 1 ft = 0.305 m
No. of constraints = No. of B.C. + No. of mesh points

TABLE 2 Results for Displacement for Beam on Elastic Foundation Subject to Uniform Load by using Hermitian Shape Function

Shape function	No. of elements	No. of unknowns	No. of mesh points	No. of inter-element continuity conditions	No. of constraints	Error (%)	
						Max	Min
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
H_{ij}^{2}	1	2	3		3	0.03	0.00
	2	5	4	1	5	-0.02	-0.01
	2	5	6	1	7	-0.01	0.00
	3	8	2	2	9	0.01	0.01
H_{ij}^{3}	1	4	4		4	-0.01	0.00
	1	4	5		5	0.00	0.00
	2	8	8		8	0.00	0.00

TABLE 3 Results for the Torsion of a Prismatic Bar with Square Cross Section

Location of mesh points (1)	No. of un-knowns (2)	No. of mesh points (3)	K (4)	K_1 (5)	Remarks (6)

(a) With one element for a quadrant

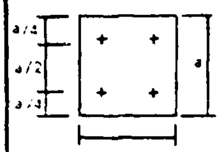
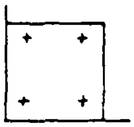
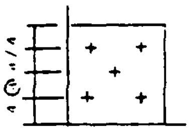
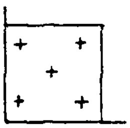
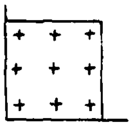
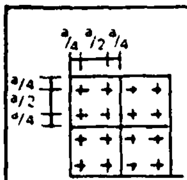
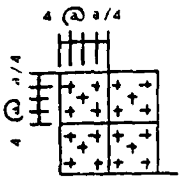
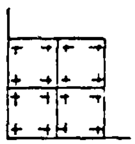
	4	4	0.651	0.1389	
	4	4	0.670	0.1432	2x2 Gaussian points
	4	5	0.636	0.1356	
	4	5	0.648	0.1383	2x2 Gaussian points plus center
	4	9	0.674	0.1454	3x3 Gaussian points

TABLE 3 Continued

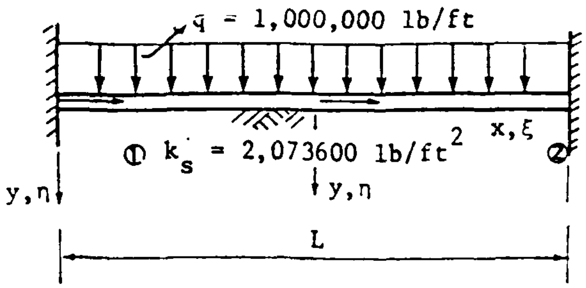
(b) With 4 elements for a quadrant

	16	16	0.672	0.1389	
	16	20	0.669	0.1391	
	16	16	0.675	0.1408	2x2 Gaussian points

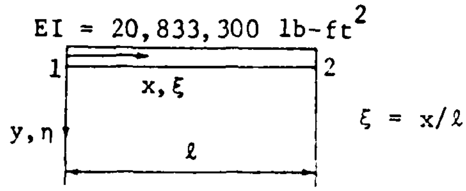
(c) Exact (32)

			0.675	0.1406	

Note: a = 5 in (127 mm)
+ represents a mesh point



(a) Beam with two elements



(b) Beam element

(1 ft. = 0.305 m; 1 lb/ft = 14.59 N/m; 1 lb/ft^2 = 47.837 N/m^2; 1 lb-ft^2 = 0.414 N-m^2)

Fig. 1 Beam on Elastic Foundation

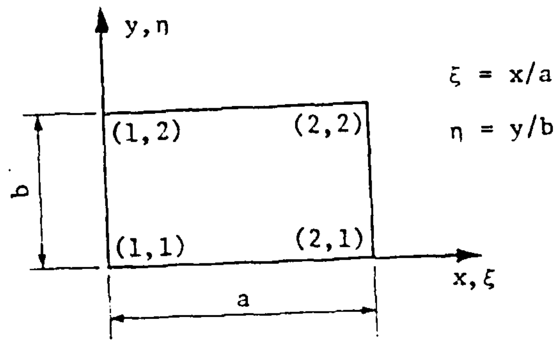
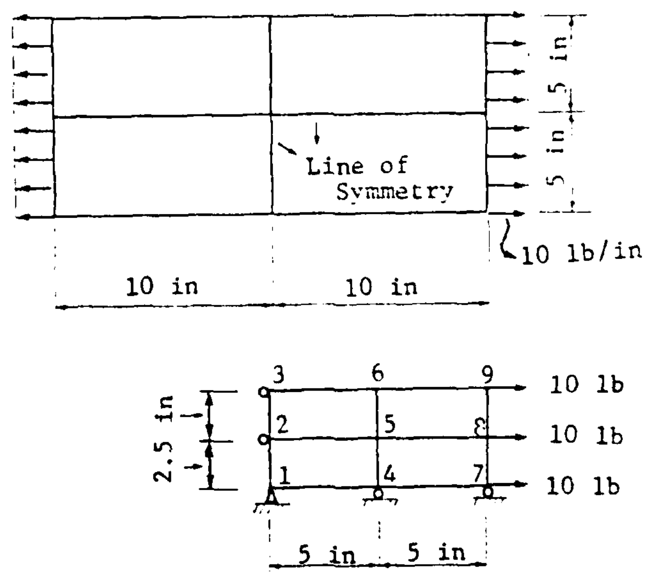
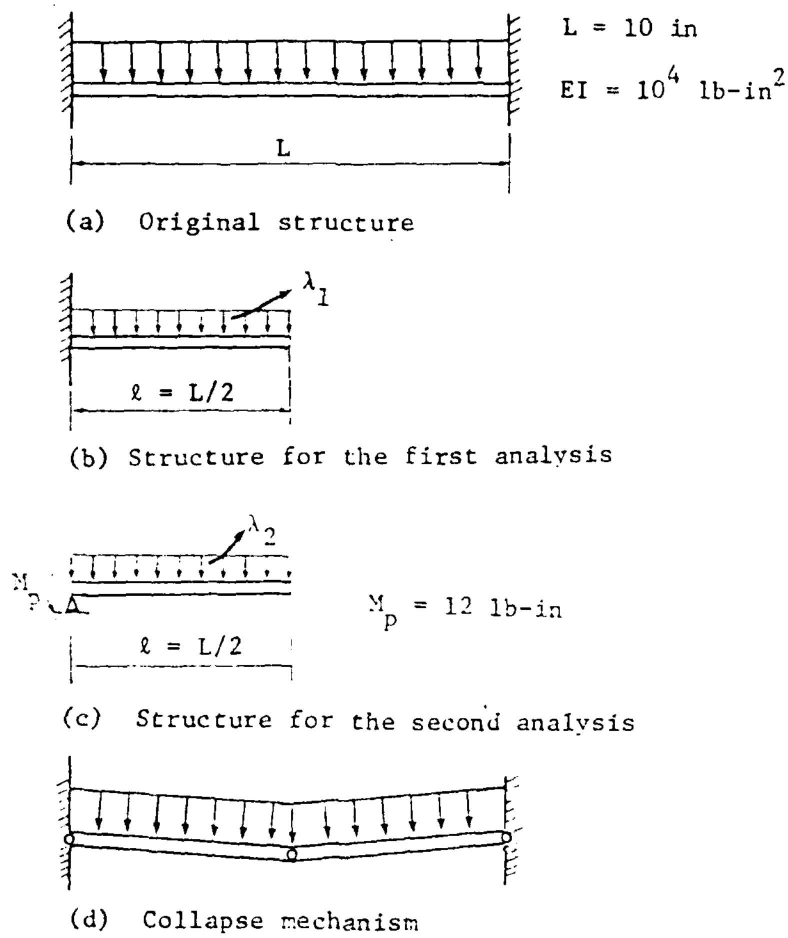


Fig. 2 Rectangular Element



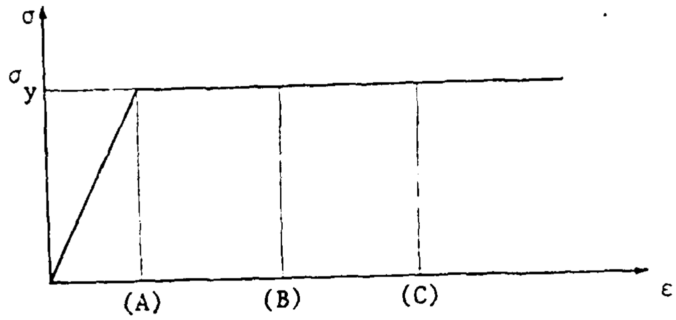
(1 in = 25.4 mm; 1 lb = 4.45 N; 1 lb/in = 0.175 N/mm)

Fig. 3 Pure Tension

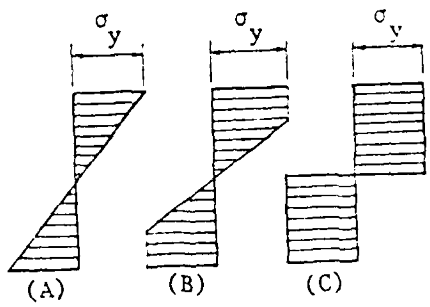


(1 in = 25.4 mm; 1 lb-in^2 = 2870.692 N-mm^2; 1 lb-in = 113.03 N-mm)

Fig. 2 Limit Analysis of a Fixed-fixed Beam



(a) Stress-strain relationship for extreme fiber



(b) Stress distribution

Fig. 5 Plastic Bending

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